Algebraic Number Theory Final Exam

July 2015

This exam is of 50 marks. There are 5 questions, each of 10 marks. Some of them have sub-parts. Please read all the questions carefully and do not cheat. Good luck! (50)

1. Minkowski's bound implies that in every ideal class [I] there is an any ideal $J \in [I]$ with

$$N(J) \leqslant \left(\frac{4}{\pi}\right)^s \cdot \frac{n!}{n^n}\sqrt{|\Delta_K|}.$$

where Δ_K is the discriminant and $n = r + 2s = [K : \mathbb{Q}]$. Use it to compute the class number of
a. $\mathbb{Q}(\sqrt{7})$. (5)
b. $\mathbb{Q}(\sqrt{-5})$. (5)

2a. Let $K = \mathbb{Q}(\sqrt{-13})$. Obtain congruences conditions to determine whether a prime in $\mathbb{Z}$ splits completely, ramifies or remains inert in the ring of integers of K. (8)

2b. Using (a) determine what happens to the prime 23. (2)

3a. Let $\mathbb{U}_p$ denote the group of units in $\mathbb{Z}_p$, the valuation ring of $\mathbb{Q}_p$. Recall that

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \| x|_p \leqslant 1\}.$$

Show that (5)

$$\mathbb{U}_p = \{x \in \mathbb{Q}_p \| x|_p = 1\}$$

3b Show that $\mathbb{Q}_p^* = \mathbb{Q}_p - \{0\}$ satisfies (5)

$$\mathbb{Q}_p^* \simeq \mathbb{Z} \times \mathbb{U}_p$$

4. Prove that $\mathbb{Z}_p$ is compact with respect to the metric topology induced by the p-adic metric. (Hint: One way to do it is to show it is **complete** and **totally bounded**, where totally bounded means that it can be covered by finitely many ϵ-balls for any $\epsilon > 0$.) (10)

5a. Show that is $\xi = \exp(2\pi i/m)$ and $(k, m) = 1$ then

$$\frac{(1-\xi^k)}{(1-\xi)}$$

is a unit in $\mathbb{Z}[\xi]$ (5)

5b. Show (5)

$$\log(1-\xi^k) = \log(2) + \log(\sin(\frac{k\pi}{m})) + \frac{k-1}{2}\pi i$$